BENCH MARK

January 2017 issue . . .

- Higher Order Methods
- Particle Methods
- Virtual Engineering to Ensure Excellent Customer Experiences for Life
- Introduction to LOTAR of Engineering Analysis and Simulation Data
- What is Equilibrium Finite Element Analysis
- Unfitted Finite Element Methods: a Modern Paradigm for Advanced Numerical Simulations
- NURBS-Enhanced Finite Element Method: Towards a Seamless CAD-FEA Integration
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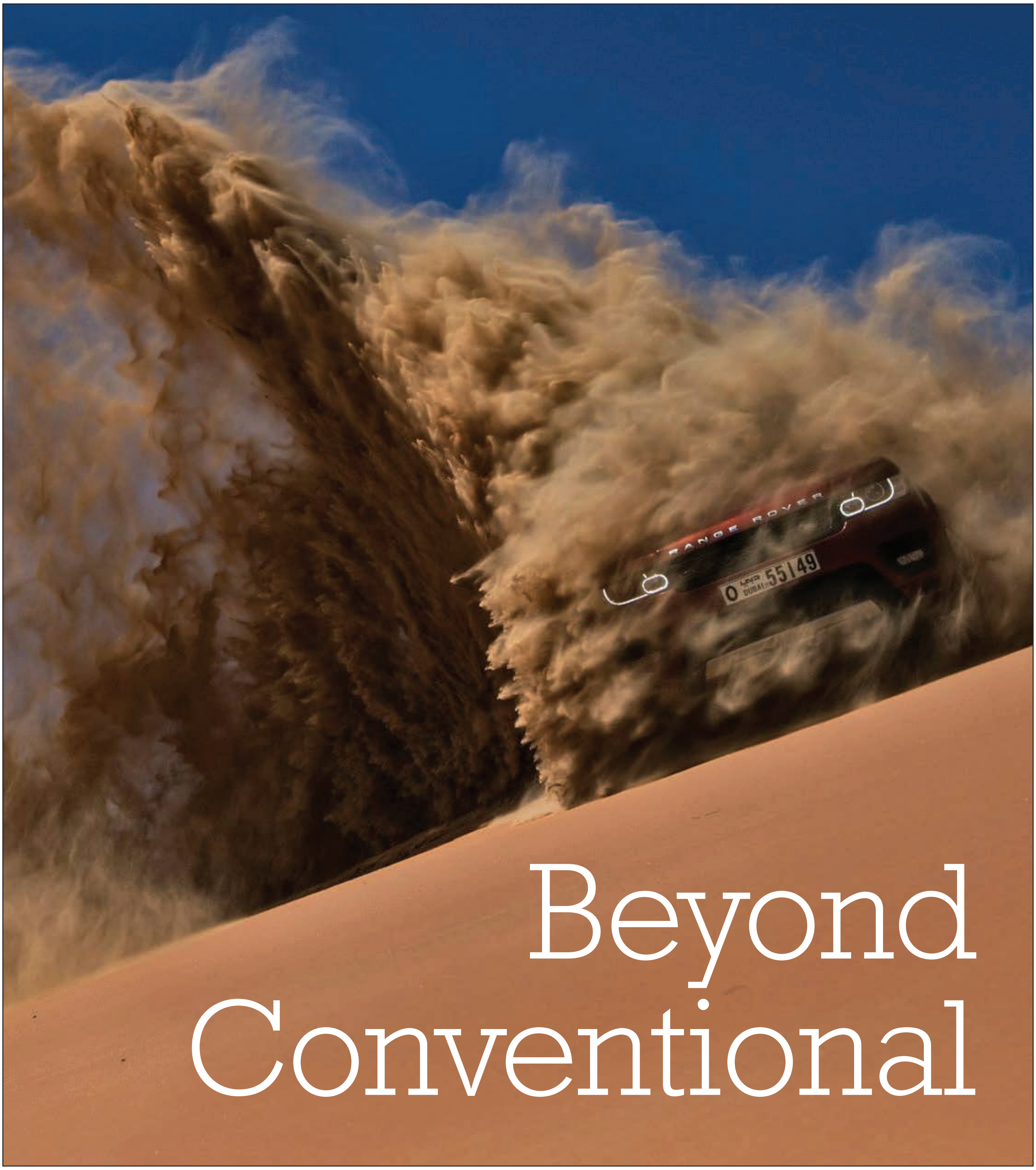

History and Use of High Order Finite Element Methods in Professional Practice

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Whenever engineering decisions are based on the results of numerical simulation there is an implied expectation of reliability. Without such expectation it would not be possible to justify the time and cost of a simulation project. If simulation produces misleading information then it has a negative economic value with possibly severe consequences. There are many well documented instances of expensive repairs, retrofits, project delays and serious safety issues arising from lack of quality assurance in numerical simulation. A new NAFEMS publication addresses the importance of credibility in numerical simulation from the perspective of management [1].

Milestones

To help understand the relationship between conventional and high order finite element methods and the role of high order methods in establishing the credibility of numerical simulation, we review some of the major milestones in the development of the finite element method (FEM). These milestones are numbered in Figure 1. Brief descriptions follow.

1. The first paper on FEM was published in 1956. In the following year the Soviet Union launched the first satellite, Sputnik, and the space race begun. This brought about substantial investments into engineering and scientific projects in support of the US space program.

2. NASA issued a request for proposal that eventually led to the development of the finite element software NASTRAN. This marks the beginning of the development of legacy codes. The infrastructure of these codes was established during the following 5 to 7 years.

3. Early work on the FEM was performed by engineers. The first mathematical papers were published in 1972. This is an important milestone because mathematicians view FEM very differently from engineers: Engineers think of FEM as a modeling tool that permits joining various elements selected from a finite element library. They believe that the solution of the corresponding numerical problem approximates the physical response of their object of interest such as an airframe, turbine disk, pressure vessel, etc. to loads or other excitations.

   Mathematicians, on the other hand, view FEM as a method by which approximate solutions can be obtained to mathematical problems. For example, the equations of linear elasticity, together with the solution domain, the material properties, the loading conditions and constraint conditions define a mathematical problem that has a unique exact solution $\mathbf{u}_{EX}$. The finite element solution $\mathbf{u}_{FE}$ is an approximation to $\mathbf{u}_{EX}$.

   A key question is; how close are the quantities of interest (QoI) corresponding to $\mathbf{u}_{EX}$ to the QoI corresponding to $\mathbf{u}_{FE}$? The answer to this question is obviously important to engineers. Relying on the QoI corresponding to $\mathbf{u}_{FE}$ without having some estimate of the size of the error of approximation would be dangerous practice. Unfortunately, finite element models often do not correspond to a well-defined mathematical problem and in those cases this error cannot be estimated. It is not even defined.

   The accuracy of approximation depends on the finite element mesh and the polynomial degree of the elements. In the early implementations of the finite element method the polynomial degree of the elements (denoted by p) was fixed at a low value, typically p=1 or p=2, and the error of approximation was controlled by mesh refinement such that the size of the largest element in the mesh, denoted by h, was reduced. This is known as the h-version of the finite element method.

   In the mid-1970s research indicated that keeping the finite element mesh fixed and increasing p, has important advantages. This is known as the p-version.

4. It was proven and demonstrated in 1981 for a large class of problems, which includes elasticity, that the rate of convergence of the p-version measured in the energy norm is at least twice that of the h-version.

5. It was proven and demonstrated in 1984 that when the finite element mesh is properly graded then the error of approximation goes to zero exponentially as p is increased. The precise statement is this:

$$\sqrt{U(\boldsymbol{u}_{EX} - \boldsymbol{u}_{FE})} \approx \frac{k}{e^{\gamma N^{\theta}}}$$

   where U is the strain energy, k, γ, θ are parameters that depend on $\mathbf{u}_{EX}$. Therefore the h- and p-versions are special cases of the finite element method where both the mesh and the polynomial degree are important in controlling the error of approximation.

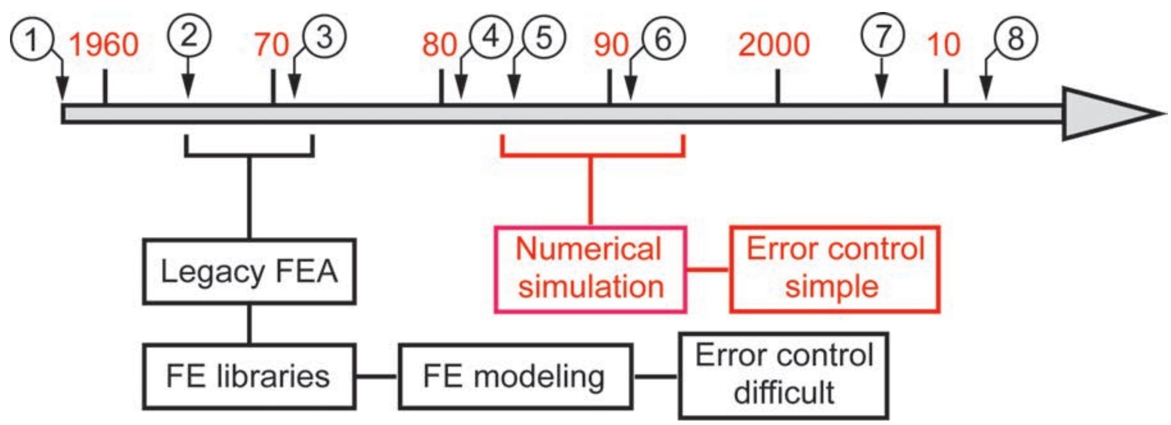

Figure 1: FEM development timeline

The distinction between the h- and p-versions of the FEM is rooted in the history of the development of FEM rather than in its theoretical foundations.

The following example, a detailed description of which is available in [2], illustrates the striking differences in convergence rates among the h-version, with uniform mesh refinement, the p-version on a uniform mesh and the p-version on an optimally graded mesh. This is a problem of planar elasticity constructed such that the exact strain energy is known, therefore the error of approximation in energy norm is known.

The relative error in energy norm is plotted against the number of degrees of freedom (N) on log-log scale in Figure 2 for uniformly refined meshes and an optimally graded mesh shown in the lower left corner.

We see that to achieve 1% relative error we need approximately 10^3 degrees of freedom for the p-version on the optimally graded mesh whereas we need about 10^7 degrees of freedom for the h-version (with p = 2) using uniformly refined meshes. Taking the sparsity of the matrices into consideration, let us assume that the operation count is proportional to $N^{3/2}$. Then the h-version with uniform mesh refinement will require one million times as many operations as the p-version on an optimal mesh.

The most important practical advantage of the p-version is that it makes estimation and control of the accuracy of computed information much more efficient and more convenient than the h-version.

6. Any mathematical model can be viewed as a special case of a more comprehensive model. Therefore mathematical models are members of a hierarchic structure. For example, a model based on the assumptions of the linear theory of elasticity is a special case of a model that accounts for plastic deformation. Once the solution of a problem of linear elasticity is available, it is possible to test whether the assumptions incorporated in the model are satisfied. If the answer is no then the analyst needs to employ a higher model. The implementation of a hierarchic modeling framework begun in 1991. It provides for seamless transition from lower to higher models. As indicated in Figure 1, the development and documentation of hierarchic models was substantially completed by the mid-1990s.

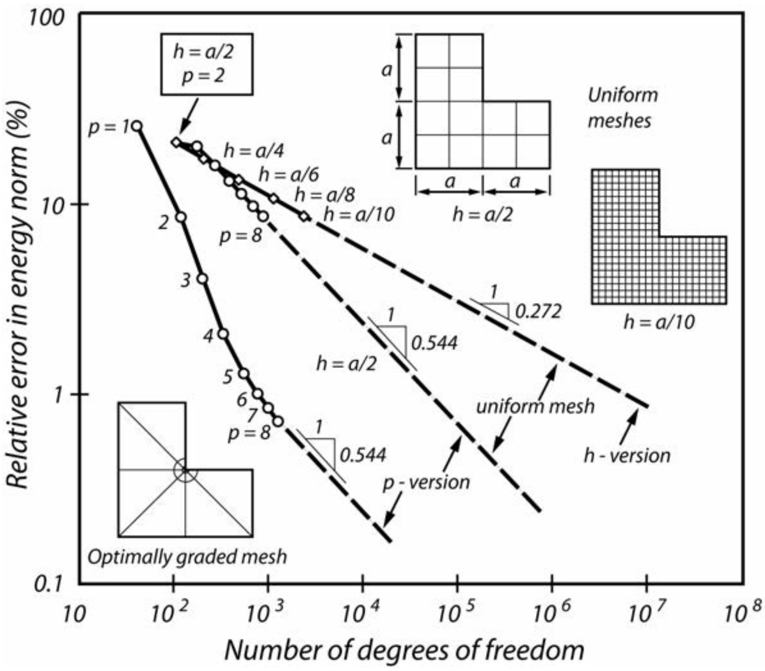

Figure 2: Convergence of the h- and p-versions.

7. The American Society of Mechanical Engineers (ASME) published its first guideline on verification and validation (V&V) in computational solid mechanics in 2006. The main point is this: Since engineering decisions are based on computed information, assurance of the quality of that information is essential. Specifically, the following quality control steps are recommended by ASME: (a) code verification, (b) estimation of numerical solution error with respect to each of the QoIs, and (c) validation of the simulation results by comparison with available experimental data.

   Verification and validation are major aspects of the work of the Analysis Management Working Group (AMWG) of NAFEMS which is currently developing a range of publications on this theme.

8. The concept of simulation governance from the perspective of mechanical and structural engineering was introduced in 2012 [3]. Simulation governance is a managerial function concerned with the exercise of command and control over all aspects of numerical simulation through the establishment of processes for the systematic improvement of the tools of engineering decision making. This includes: (a) proper formulation of mathematical models, (b) selection and adoption of the best available numerical simulation technology, (c) coordination of experimental work with numerical simulation, (d) documentation and archival of experimental data, (e) application of data and solution verification procedures, (f) revision of mathematical models in the light of new information collected from physical experiments and field observations and (g) standardization of design, analysis and certification workflows wherever appropriate.

Solution verification

Solution verification is an essential technical requirement in numerical simulation. It is concerned with the question of how close are the quantities of interest (QoI) corresponding to $\mathbf{u}_{FE}$ to the QoI corresponding to $\mathbf{u}_{EX}$? In general we do not know $\mathbf{u}_{EX}$, however we do know that any QoI corresponding to $\mathbf{u}_{EX}$ is independent of the method of approximation being used. Therefore if a computed number changes significantly with mesh refinement, or increase in the polynomial degree, then it cannot be accurate.

To illustrate this we consider a rod end made of an aluminum alloy, with beryllium copper bearing. In this example the QoI is the stress concentration factor, defined as the ratio of the maximum normal stress to the average tensile stress in the shaft.

Using the mesh shown in Figure 3, we increase the polynomial degree of the elements from 1 to 8 and plot the computed stress concentration factor. It is seen that the data points corresponding to p = 3 to p = 8 change very little. In retrospect we could have stopped at p = 3 and obtained a very close approximation. However the accuracy of the solution would not have been verified because we would not have known that any further increase in the number of degrees of freedom would not affect our estimate of the stress concentration factor significantly.

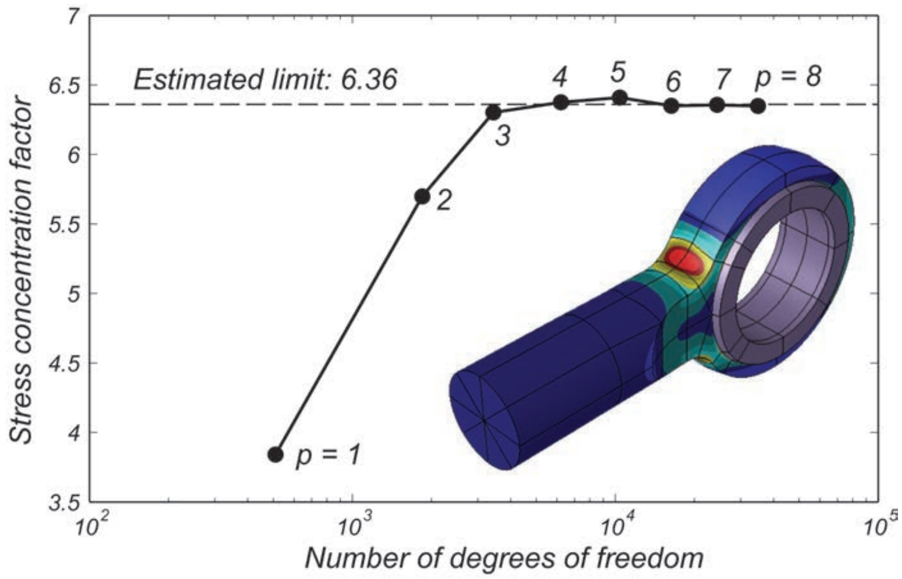

Figure 3: Illustration of solution verification by the p-version.

Large aspect ratios and robustness

The p-version tolerates elements with large aspect ratios. This is especially important when analyzing laminated composites where ply-by-ply representation is necessary for resolving local stress and strain distributions. Elements with large aspect ratios are also necessary in finite element analyses of plate and shell problems.

Another important advantage of the p-version is that its performance is much less sensitive to input data than the h-version. For example, if Poisson's ratio is close to 0.5 then the h-version exhibits a highly undesirable property, known as Poisson's ratio locking. The p-version is insensitive to Poisson's ratios.

Outlook

Now that solution verification is an established technical requirement, it is expected that the practical utility of high order methods will be increasingly recognized by the engineering community. These methods have been successfully used in the aerospace/defense sector since the 1980s.

Another motivator for wider use of high order methods is a growing interest in the democratization of FEA, that is, the creation and deployment of smart applications by expert FEA analysts for safe and routine use by engineers and designers who are not required to have the expertise to properly formulate the problems themselves. High order methods are ideally suited for supporting the autonomous error control procedures that smart applications must provide. ■

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He is a founding member and Fellow of the US Association for Computational Mechanics. Among his honors are election to the Hungarian Academy of Sciences as External Member and an honorary doctorate.

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